

Generalization of the Gell-Mann formula for $sl(n, \mathbb{R})$ and $su(n)$ algebras

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Abstract

The so called Gell-Mann or decontraction formula is proposed as an algebraic expression inverse to the Inönü-Wigner Lie algebra contraction. It is tailored to express the Lie algebra elements in terms of the corresponding contracted ones. In the case of $sl(n, \mathbb{R})$ and $su(n)$ algebras, contracted w.r.t. $so(n)$ subalgebras, this formula is generally not valid, and applies only in the cases of some algebra representations. A generalization of the Gell-Mann formula for $sl(n, \mathbb{R})$ and $su(n)$ algebras, that is valid for all tensorial, spinorial, (non)unitary representations, is obtained in a group manifold framework of the $SO(n)$ and/or $Spin(n)$ group. The generalized formula is simple, concise and of ample application potentiality. The matrix elements of the $\overline{SL}(n, \mathbb{R})/Spin(n)$, i.e. $SU(n)/SO(n)$, generators are determined, by making use of the generalized formula, in a closed form for all irreducible representations.

1 Introduction

Relations between various algebras/groups as well as their subalgebras/subgroups played an important role in the Lie algebras/groups theory and its representation theory development. One of these relations is the well known Inönü-Wigner contraction [1] that was, besides its mathematical merits, an important tool in numerous physical applications. There is a variety of Inönü-Wigner Lie algebra contraction applications arising in various parts of Theoretical Physics. Just to mention a few ranging from contractions from the Poincaré algebra to the Galilean one, and from the Heisenberg algebras to the Abelian ones of the same dimensions (a symmetry background of a transition processes from relativistic and quantum mechanics to classical mechanics) to those of contractions in the Kaluza-Klein gauge theories framework, say from (Anti-)deSitter to the Poincaré algebra, and various cases involving the Virasoro and Kac-Moody algebras. For instance, our recent study of the Affine Gauge Gravity Theory in $5D$ [2] is heavily related to the $sl(5, \mathbb{R})$ algebra contraction w.r.t. its $so(1, 3)$ subalgebra, and the representations of the relevant algebras. In physical terms, the meaning of the Inönü-Wigner contraction is to relate, for instance, strong to weak coupling regimes of the corresponding theories, or geometrically curved to “less curved” and/or flat spaces.

It is not so well known that there is an inverse to the Inönü-Wigner contraction called the Gell-Mann formula [3, 4, 5, 6]. This formula is a simple prescription designed to determine a deformation of a Lie algebra that is “inverse” to the Inönü-Wigner contraction. This formula relates elements of the starting algebra to the corresponding ones of the contracted algebra. Moreover, by its construction it relates also the representations of these two algebras. Since, by a rule, various properties of the contracted algebra are much easier to explore (e.g. construction of representations [7], decompositions of a direct product of representations [4], etc.), this formula found its place, as a useful and simple tool, even in some textbooks and in the mathematical encyclopedia [3].

To emphasize its meaning, this formula is also referred to as the “decontraction” formula. This formula was introduced by Dothan and Ne’eman (while working at Caltech on noncompact algebras and their representations) [8] and advocated by Hermann who learned from Gell-Mann about this formula, named it after Gell-Mann, and made important contributions to the formula himself.

In the general case the Gell-Mann formula construction goes as follows: Let \mathcal{A} be a symmetric Lie algebra $\mathcal{A} = M + T$ with a subalgebra \mathcal{M} such that:

$$(1) \quad [M, M] \subset M, \quad [M, T] \subset T, \quad [T, T] \subset M.$$

Further, let \mathcal{A}' be its Inönü-Wigner contraction algebra w.r.t its subalgebra \mathcal{M} , i.e. $\mathcal{A}' = M + U$, where

$$(2) \quad [M, M] \subset M, \quad [M, U] \subset U, \quad [U, U] = \{0\}.$$

The Gell-Mann formula states that the elements $T \in \mathcal{T}$ can be in certain cases expressed in terms of the contracted algebra elements $M \in \mathcal{M}$ and $U \in \mathcal{U}$ by the following rather simple expression:

$$(3) \quad T = i \frac{\alpha}{\sqrt{U \cdot U}} [C_2(\mathcal{M}), U] + \sigma U.$$

Here, $C_2(\mathcal{M})$ and $U \cdot U$ denote the second order Casimir operators of the \mathcal{M} and \mathcal{A}' algebras respectively, while α is a normalization constant and σ is an arbitrary parameter. For a mathematically strict definition, cf. [3].

The main drawback of the Gell-Mann formula is its limited validity. There is a number of references dealing with the question when this formula is applicable [4, 5, 6, 9]. The formula is best studied in the case of (pseudo) orthogonal algebras $so(m, n)$ contracted w.r.t. their $so(m-1, n)$ and/or $so(m, n-1)$ subalgebras, i.e. in the corresponding group cases: $SO(m, n) \rightarrow R^{m+n-1} \wedge SO(m-1, n)$ and/or $SO(m, n) \rightarrow R^{m+n-1} \wedge SO(m, n-1)$, where, loosely speaking, the Gell-Mann formula works very well [10]. Moreover, the case of (pseudo) orthogonal algebras is the only one where this formula is valid for (almost) all representations [11]. Recently, we studied the $sl(n, \mathbb{R})$ cases where the Gell-Mann formula does not hold as a general operator expression and its validity depends heavily on the $sl(n, \mathbb{R})$ algebra representation space. An exhaustive list of the cases for which the Gell-Mann formula for $sl(n, \mathbb{R})$ algebras hold is obtained [9].

There were some attempts to generalize the Gell-Mann formula for the “decontracted” algebra operators of the complex simple Lie algebras g with respect to decomposition $g = k + ik = k_c$ [12, 13], that resulted in a form of relatively complicated polynomial expressions.

In this work we consider Gell-Mann’s formula in the $sl(n, \mathbb{R})$ algebra cases, where the contraction is performed w.r.t. their maximal compact $so(n)$ subalgebras. The Gell-Mann formula in these cases is especially valuable as a tool in the problem of finding all unitary irreducible representations of the $sl(n, \mathbb{R})$ algebras in the basis of the $SO(n)$ and/or $Spin(n)$ groups generated by their $so(n)$ subalgebras. Finding representations in the basis of the

maximal compact subgroup $SO(n)$ of the $SL(n, \mathbb{R})$ group, is mathematically superior, and it suites well various physical applications in particular in nuclear physics, gravity, physics of p-branes [14] etc. As an example consider a gauge theory based on the Affine spacetime symmetry $SA(n, \mathbb{R}) = T_n \wedge \overline{SL}(n, \mathbb{R})$. The gauge covariant derivative, D_μ , $\mu = 0, 1, \dots, n-1$, as acting on an Affine matter field $\Psi(x)$, is given by,

$$D_\mu \Psi(x)_i = \left(\partial_\mu - i\Gamma_\mu^{ab}(x) (Q_{ab})_i^j \right) \Psi(x)_j, \quad Q_{ab} \in sl(n, \mathbb{R}),$$

where $\Gamma_\mu^{ab}(x)$ are the $sl(n, \mathbb{R})$ connections, and i, j enumerate the matter field components. The matter-gravity vertices require the knowledge of the $sl(n, \mathbb{R})$ operators matrix elements $(Q_{ab})_i^j$ in the Hilbert space of the matter field components $\{\Psi_i(x)\}$. In particular, for a generic spinorial $\overline{SL}(n, \mathbb{R})$ matter field, an explicit form of the matrix elements of the $sl(n, \mathbb{R})$ generators for infinite-dimensional representation corresponding to the Ψ field is required.

Moreover, this framework opens up a possibility of finding, in a rather straightforward manner, all matrix elements of noncompact $SL(n, \mathbb{R})$, i.e. $\overline{SL}(n, \mathbb{R})$ generators for all finite and infinite dimensional representations. Unfortunately, as stated above, the original Gell-Mann formula is of limited validity, and it can be applied in the classes of multiplicity free representations only. Recently we have demonstrated, by an explicit construction, that the Gell-Mann formula has a generalization that is valid for all irreducible representations of the $sl(n, \mathbb{R})$, $n = 3, 4$ and 5 algebras [15]. These formulas, though comparatively simpler than the ones resulting from some other attempts to generalize the Gell-Mann formula, are still rather cumbersome taking a few rows. In this paper we present a generalization of the Gell-Mann formula that is almost as compact and simple as the original formula, which is valid for *all representations of the $sl(n, \mathbb{R})$ algebras for all n values*.

In our previous study of the Gell-Mann formula and its generalization for the $sl(n, \mathbb{R})$, $n = 3, 4, 5$ algebras [15], we first extracted the generalized formula from the previously known generic expressions in the $sl(3, \mathbb{R})$ and $sl(4, \mathbb{R})$ cases which enabled us to obtain finally the generalized formula for the $sl(5, \mathbb{R})$ algebra. A crucial ingredient in generalization of the Gell-Mann formula for the $sl(n, \mathbb{R})$ algebras is a role of the “left-rotation” action of the $Spin(n)$ subgroup of the corresponding $\overline{SL}(n, \mathbb{R})$ group, which manages the non trivial multiplicity of the maximal compact $so(n)$ subalgebra representations for a generic $sl(n, \mathbb{R})$ representation. In this work we show, by making use of the Cartesian basis, how to obtain the explicit form of the generalized Gell-Mann formula for an arbitrary $sl(n, \mathbb{R})$ algebra with a left-rotations included properly and demonstrate the closure of the $sl(n, \mathbb{R})$

algebra commutation relations. Moreover, we showed how to rewrite the expression of the generalized formula in a suitable basis (e.g. the Gel'fand-Tsetlin basis) allowing us to directly write down matrix expressions of the $sl(n, \mathbb{R})$, i.e. $su(n)$, generators for an arbitrary irreducible representation in the basis of the $Spin(n)$ group.

Note that due to mutual relations between the $sl(n, \mathbb{R})$ and $su(n)$ algebras, one can convey the Gell-Mann formula results obtained for the $sl(n, \mathbb{R})$ algebras to the corresponding ones of the $su(n)$ algebras. Though, there are some subtleties in that process that are considered below.

2 $sl(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ topology and representations considerations

As already stated, the Gell-Mann formula, except in the cases of (pseudo) orthogonal algebras, is not generally valid by itself, and its validity depends on the representations of the algebra as well. Therefore, in the case of the $SL(n, \mathbb{R})$ groups, i.e. their $sl(n, \mathbb{R})$ algebras, one faces, in addition to the pure algebraic features, the matters that are relevant to the group/algebra representations theory as well: notably (i) the group topology properties, and (ii) the non trivial multiplicity of the $SL(n, \mathbb{R})$, and $sl(n, \mathbb{R})$ representations in the $SO(n)$, and $so(n)$ basis, respectively. Both features are rather subtle for $n \geq 3$. Note that, in the case of the $sl(n, \mathbb{R})$ algebras, due to a fact that the generalization of the Gell-Mann formula depends on the algebra irreducible representation features, the construction itself deviates from the standard Lie algebra deformation approach.

The $SL(n, \mathbb{R})$ group can be decomposed, as any semisimple Lie group, into the product of its maximal compact subgroup $K = SO(n)$, an Abelian group A and a nilpotent group N . It is well known that only K is not guaranteed to be simply-connected. There exists a universal covering group $\overline{K} = \overline{SO}(n)$ of $K = SO(n)$, and thus also a universal covering of $G = SL(n, \mathbb{R})$: $\overline{SL}(n, \mathbb{R}) \simeq \overline{SO}(n) \times A \times N$. For $n \geq 3$, $SL(n, \mathbb{R})$ has double covering, defined by $\overline{SO}(n) \simeq Spin(n)$ the double-covering of the $SO(n)$ subgroup. The universal covering group \overline{G} of a given group G is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering, $\overline{SL}(n, \mathbb{R})$ exists provided one can embed $\overline{SL}(n, \mathbb{R})$ into a group of finite complex matrices that contain $Spin(n)$ as a subgroup. A scan of the Cartan classical algebras points to the $SL(n, \mathbb{C})$ groups as a natural candidate for the $SL(n, \mathbb{R})$ groups covering. However, there is no match of the defining dimensionalities of the $SL(n, \mathbb{C})$ and $Spin(n)$ groups for $n \geq 3$, $dim(SL(n, \mathbb{C})) = n < 2^{\lfloor \frac{n-1}{2} \rfloor} = dim(Spin(n))$, except for $n = 8$. In the

$n = 8$ case, one finds that the orthogonal subgroup of the $SL(8, \mathbb{R})$ and $SL(8, \mathbb{C})$ groups is $SO(8)$ and not $Spin(8)$. For a detailed account of the $D = 4$ case cf. [16]. Thus, we conclude that there are no covering groups of the $SL(n, \mathbb{R})$, $n \geq 3$ groups defined in finite-dimensional spaces. An explicit construction of all $SL(3, R)$ irreducible representations, unitary and nonunitary multiplicity-free spinorial [17], and unitary non-multiplicity-free [18], shows that they are infinite-dimensional. The universal (double) covering groups, $\overline{SL}(n, \mathbb{R})$, $n \geq 3$ of the $SL(n, \mathbb{R})$, $n \geq 3$ group are groups of infinite complex matrices. All their spinorial representations are infinite dimensional. In the reduction of this representations w.r.t. $Spin(n)$ subgroups, one finds $Spin(n)$ representations of unbounded spin values.

The $SU(n)$ groups are compact, with a simply connected group manifold, thus being its own universal coverings. The $SO(n)$ subgroups are embedded into the $SU(n)$ groups as n -dimensional matrices, and this embedding does not allow nontrivial (double) covering of $SO(n)$ within $SU(n)$. As a consequence, in the reduction of the $SU(n)$ unitary irreducible representations one finds the tensorial $SO(n)$ representations only.

An inspection of the unitary irreducible representations of the $\overline{SL}(n, \mathbb{R})$, $n = 3, 4$ groups [18, 19] shows that they have, as a rule, a nontrivial multiplicity of the $Spin(n)$, $n = 3, 4$ subgroup representations. It is well known, already from the case of the $SU(3)$ representations in the $SO(3)$ subgroup basis, that the additional labels required to describe this nontrivial multiplicity cannot be solely related to the group generators themselves. An elegant solution, that provides the required additional labels, is to work in the group manifold of the $SO(n)$ maximal compact subgroup, and to consider an action of the group both to the right and to the left. In this way one obtains, besides the maximal compact subgroup labels, an additional set of labels to describe the $SO(n)$ subgroup multiplicity.

3 Inönü-Wigner contraction of $sl(n, \mathbb{R})$ algebras

The $sl(n, \mathbb{R})$ algebra operators, i.e. the $SL(n, \mathbb{R})$, $\overline{SL}(n, \mathbb{R})$ group generators, can be split into two subsets: M_{ab} , $a, b = 1, 2, \dots, n$ operators of the maximal compact subalgebra $so(n)$ (corresponding to the antisymmetric real $n \times n$ matrices, $M_{ab} = -M_{ba}$), and the, so called, sheer operators T_{ab} , $a, b = 1, 2, \dots, n$ (corresponding to the symmetric traceless real $n \times n$ matrices,

$T_{ab} = T_{ba}$). The $sl(n, \mathbb{R})$ commutation relations, in this basis, read:

$$(4) \quad [M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}),$$

$$(5) \quad [M_{ab}, T_{cd}] = i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}),$$

$$(6) \quad [T_{ab}, T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}).$$

The $su(n)$ algebra operators can be split likewise w.r.t. its $so(n)$ subalgebra into M_{ab} and $T_{ab}^{su(n)}$, $a, b = 1, 2, \dots, n$. The $T_{ab}^{su(n)}$ and T_{ab} operators are mutually related by $T_{ab}^{su(n)} = i T_{ab}$, and the $[T_{ab}^{su(n)}, T_{cd}^{su(n)}]$ differs from (6) by having an overall plus sign on the right-hand side.

The Inönü-Wigner contraction of $sl(n, \mathbb{R})$ with respect to its maximal compact subalgebra $so(n)$ is given by the limiting procedure:

$$(7) \quad U_{ab} \equiv \lim_{\epsilon \rightarrow 0} (\epsilon T_{ab}),$$

which leads to the following commutation relations:

$$(8) \quad [M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca})$$

$$(9) \quad [M_{ab}, U_{cd}] = i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca})$$

$$(10) \quad [U_{ab}, U_{cd}] = 0.$$

Therefore, the Inönü-Wigner contraction of $sl(n, \mathbb{R})$ gives a semidirect sum $r_{\frac{n(n+1)}{2}-1} \ltimes so(n)$ algebra, where $r_{\frac{n(n+1)}{2}-1}$ is an Abelian subalgebra (ideal) of “translations” in $\frac{n(n+1)}{2} - 1$ dimensions.

The Gell-Mann formula, a prescription to provide an “inverse” to the Inönü-Wigner contraction, (7), in this case reads:

$$(11) \quad T_{ab}^{\sigma} = \frac{i\alpha}{\sqrt{U \cdot U}} [C_2(so(n)), U_{ab}] + \sigma U_{ab},$$

where $C_2(so(n))$ denotes the second order Casimir operator of the $so(n)$ subalgebra, $\frac{1}{2} \sum M_{ab} M_{ab}$, while σ is an arbitrary (complex) parameter and α is a (real) normalization constant that depends on n . This expression can also be written in an equivalent form:

$$(12) \quad T_{ab}^{\sigma'} = -\frac{2\alpha}{\sqrt{U \cdot U}} \sum_c U_{c\{a} M_{b\}c} + \sigma' U_{ab},$$

where σ' differs from σ accordingly, and $\{ \}$ denotes symmetrization of the enclosed indices.

In order to make use of the Gell-Mann formula to obtain the $sl(n, \mathbb{R})$ representations, the first necessary step is to determine representation matrix elements of the contracted algebra operators. The corresponding contracted

group is a semidirect product of $SO(n)$ and an Abelian group, and it is well known that the usual group induction method provides the complete set of all inequivalent irreducible representations [7]. Nevertheless, we will not pursue the induction approach here. Instead, we will rather proceed to work in the representation space of square integrable functions $\mathcal{L}^2(Spin(n))$ over the $Spin(n)$ group (in accord with the $SL(n, \mathbb{R})$ topological properties), with the standard invariant Haar measure. As for our final goal, this approach ensures certain advantages: (i) The generalized Gell-Mann formula is expressed in terms of tensor operators w.r.t. the maximal compact subgroup basis (instead w.r.t. the eigenvector basis of the Abelian subgroup), (ii) This representation space allows for all inequivalent irreducible representations of the contracted group (some of the irreducible representations are multiply contained, i.e. each such representation appears as many times as is the dimension of the corresponding little group representation and all of them, irrespectively of the corresponding stabilizer, can be treated in an unified manner), and (iii) this space is rich enough to contain all representatives from equivalence classes of the $\overline{SL}(n, \mathbb{R})$ group, i.e. $sl(n, \mathbb{R})$ algebra representations [20]. The last feature provides the necessary requirement of a framework needed for generalization of the Gell-Mann formula, i.e. a unique framework providing for all $sl(n, \mathbb{R})$ (unitary) irreducible representations.

The generators of the contracted group are generically represented in this space as follows. The $so(n)$ subalgebra operators act, in a standard way, via the group action to the right:

$$M_{ab} |\phi\rangle = -i \frac{d}{dt} \exp(itM_{ab}) \Big|_{t=0} |\phi\rangle, \quad g' |g\rangle = |g'g\rangle, \quad |\phi\rangle \in \mathcal{L}^2(Spin(n)).$$

The Abelian operators U_{ab} act multiplicatively as Wigner's D -functions (the appropriate representation $SO(n)$ group matrix elements as functions of the group parameters):

$$(13) \quad U_{ab} \rightarrow |u| D_{v(ab)}^{\square\square}(g^{-1}) \equiv \left\langle \begin{array}{c} \square\square \\ v \end{array} \middle| g^{-1} \middle| \begin{array}{c} \square\square \\ ab \end{array} \right\rangle,$$

$|u|$ being a constant norm, $g(\theta)$ being an $SO(n)$ element, and in order to simplify notation we denote by $\square\square$ (in a parallel to the Young tableaux) the symmetric second rank tensor representation of $SO(n)$. The vector $\left| \begin{array}{c} \square\square \\ ab \end{array} \right\rangle$ from the $\square\square$ representation space is determined by the ab “double” index of U_{ab} , whereas the vector $\left| \begin{array}{c} \square\square \\ v \end{array} \right\rangle$ can be an arbitrary vector belonging to the $\frac{1}{2}n(n+1) - 1$ dimensional $\square\square$ representation (the choice of v is determined, in Wigner's terminology, by the little group of the obtained representation). Taking an inverse of g in (13) ensures the correct transformation properties. The form of the representation of the Abelian operators merely reflects the

fact that they transform as symmetric second rank tensor w.r.t $so(n)$ (9) and that they mutually commute.

A natural discrete orthonormal basis in the $Spin(n)$ representation space is given by properly normalized functions of the $Spin(n)$ representation matrix elements:

$$(14) \quad \left\langle \begin{array}{c} \{J\} \\ \{k\} \{m\} \end{array} \right\rangle \equiv \int \sqrt{\dim(\{J\})} D_{\{k\}\{m\}}^{\{J\}}(g(\theta)^{-1}) d\theta |g(\theta)\rangle, \\ \left\langle \begin{array}{c} \{J'\} \\ \{k'\} \{m'\} \end{array} \middle| \begin{array}{c} \{J\} \\ \{k\} \{m\} \end{array} \right\rangle = \delta_{\{J'\}\{J\}} \delta_{\{k'\}\{k\}} \delta_{\{m'\}\{m\}},$$

where $d\theta$ is an (normalized) invariant Haar measure, $D_{\{k\}\{m\}}^{\{J\}}$ are the $Spin(n)$ irreducible representation matrix elements,

$$(15) \quad D_{\{k\}\{m\}}^{\{J\}}(\theta) \equiv \left\langle \begin{array}{c} \{J\} \\ \{k\} \end{array} \middle| R(\theta) \middle| \begin{array}{c} \{J\} \\ \{m\} \end{array} \right\rangle.$$

Here, $\{J\}$ stands for a set of the $Spin(n)$ irreducible representation labels, while $\{k\}$ and $\{m\}$ labels enumerate the $\dim(D^{\{J\}})$ representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch-Gordan coefficients of the $Spin(n)$ group as follows,

$$(16) \quad \left\langle \begin{array}{c} \{J'\} \\ \{k'\} \{m'\} \end{array} \middle| M_{ab} \middle| \begin{array}{c} \{J\} \\ \{k\} \{m\} \end{array} \right\rangle = \delta_{\{J'\}\{J\}} \sqrt{C_2(\{J\})} C_{\{m\}(ab)\{m'\}}^{\{J\}} \square_{\{J'\}}.$$

The matrix elements of the U_{ab} operators in this basis are readily found to read:

$$(17) \quad \left\langle \begin{array}{c} \{J'\} \\ \{k'\} \{m'\} \end{array} \middle| U_{ab}^v \middle| \begin{array}{c} \{J\} \\ \{k\} \{m\} \end{array} \right\rangle \\ = |u| \left\langle \begin{array}{c} \{J'\} \\ \{k'\} \{m'\} \end{array} \middle| D_{v(ab)}^{-1} \square \middle| \begin{array}{c} \{J\} \\ \{k\} \{m\} \end{array} \right\rangle \\ = |u| \sqrt{\dim(\{J'\}) \dim(\{J\})} \int D_{\{k'\}\{m'\}}^{\{J'\}*}(\theta) D_{v(ab)}^{\square}(\theta) D_{\{k\}\{m\}}^{\{J\}}(\theta) d\theta \\ = |u| \sqrt{\frac{\dim(\{J\})}{\dim(\{J'\})}} C_{\{k\} \square_v \{k'\}}^{\{J\} \square \{J'\}} C_{\{m\}(ab)\{m'\}}^{\{J\} \square \{J'\}}.$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \oplus so(n)$ representations is thus explicitly given in this space by (16) and (17).

4 The generalized formula

The Gell-Mann formula (11) expressed in terms of the Wigner's functions now reads:

$$(18) \quad T_{ab}^\sigma = i\alpha[C_2(so(n)), D_{v(ab)}^{\square\square}] + \sigma D_{v(ab)}^{\square\square}.$$

However, the T_{ab} operators, as given by this expression depending on the representation space, do not close upon the $sl(n, \mathbb{R})$ commutation relations (6) in the whole representation space. Indeed, we have shown in [9] that this formula yields the $sl(n, \mathbb{R})/so(n)$ generators only when the representation space is restricted to $Spin(n)/(Spin(m) \otimes Spin(n-m))$, with $\left| \begin{smallmatrix} \square & \square \\ v \end{smallmatrix} \right\rangle$ vector chosen to be invariant with respect to the $Spin(m) \otimes Spin(n-m)$, $1 \leq m \leq n-1$, while $Spin(1)$ is here implied to be a trivial group.

We have shown in [15] that the Gell-Mann formula can be generalized in the cases of the $sl(n, \mathbb{R})$, $n = 3, 4$ and 5 algebras so that the commutation relations (6) hold for an arbitrary irreducible representation in whole Hilbert space over the $Spin(n)$ group. This generalization was achieved by adding certain extra terms containing the generators of $SO(n)$ subgroup action to the left. For example, the generalized formula in the $sl(5, \mathbb{R})$ case reads:

$$(19) \quad \begin{aligned} T^{\sigma_1 \sigma_2 \delta_1 \delta_2}_{\substack{j_1 j_2 \\ \mu_1 \mu_2}} &= \sigma_1 D_{\substack{00 \\ 00 \mu_1 \mu_2}}^{\overline{11} j_1 j_2} + i\sqrt{\frac{1}{5}}[C_2(so(5)), D_{\substack{00 \\ 00 \mu_1 \mu_2}}^{\overline{11} j_1 j_2}] \\ &+ i \left(\sigma_2 D_{\substack{11 \\ 00 \mu_1 \mu_2}}^{\overline{11} j_1 j_2} + \frac{1}{2}[C_2(so(4)_K), D_{\substack{11 \\ 00 \mu_1 \mu_2}}^{\overline{11} j_1 j_2}] \right. \\ &- D_{\substack{11 \\ 1-1 \mu_1 \mu_2}}^{\overline{11} j_1 j_2} (\delta_1 + K_{\substack{10 \\ 00}}^{\overline{10}} - K_{\substack{01 \\ 00}}^{\overline{10}}) - D_{\substack{11 \\ -11 \mu_1 \mu_2}}^{\overline{11} j_1 j_2} (\delta_1 - K_{\substack{10 \\ 00}}^{\overline{10}} + K_{\substack{01 \\ 00}}^{\overline{10}}) \\ &\left. + D_{\substack{11 \\ 11 \mu_1 \mu_2}}^{\overline{11} j_1 j_2} (\delta_2 + K_{\substack{10 \\ 00}}^{\overline{10}} + K_{\substack{01 \\ 00}}^{\overline{10}}) + D_{\substack{11 \\ -1-1 \mu_1 \mu_2}}^{\overline{11} j_1 j_2} (\delta_2 - K_{\substack{10 \\ 00}}^{\overline{10}} - K_{\substack{01 \\ 00}}^{\overline{10}}) \right) \end{aligned}$$

The left action generators K , that appear in the formula can be related to the M_{ab} operators by the following expression:

$$(20) \quad K_{ab} \equiv g^{(a''b'')(a'b')} D_{(ab)(a''b'')}^{\square} M_{a'b'},$$

where $g^{(a''b'')(a'b')}$ is the Cartan metric tensor of $SO(n)$. The K_{ab} operators behave exactly as the rotation generators M_{ab} , it is only that they act on the lower left-hand side indices of the basis (14):

$$(21) \quad \left\langle \begin{smallmatrix} \{J'\} \\ \{k'\} \end{smallmatrix} \middle| \{m'\} \right| K_{ab} \left| \begin{smallmatrix} \{J\} \\ \{k\} \end{smallmatrix} \right| \{m\} \rangle = \delta_{\{J'\}\{J\}} \sqrt{C_2(\{J\})} C_{\{J\}}^{\{J'\}} \left| \begin{smallmatrix} \square \\ \{k\}(ab) \end{smallmatrix} \right| \{k'\} \rangle.$$

Due to the fact that the mutually contragradient $SO(n)$ representations are equivalent, the K_{ab} operators are directly related to the "left" action of the $SO(n)$ subgroup on $\mathcal{L}^2(|g(\theta)\rangle)$: $g'|g\rangle = |gg'^{-1}\rangle$. The K_{ab} and M_{ab} operators mutually commute, however, the corresponding Casimir operators match, i.e. $K_{ab}^2 = M_{ab}^2$.

The generalized Gell-Mann formulas for $sl(3, \mathbb{R})$, $sl(4, \mathbb{R})$ and $sl(5, \mathbb{R})$ [15] are given by rather cumbersome expressions. However, when these formulas are expressed in the Cartesian basis (like formulas (4)-(6)) in terms of the K_{ab} operators and anti-commutators rather than commutators the resulting expressions become extremely simple. Moreover, this form allows for an immediate generalization to the case of an arbitrary n . We prove below that the generalized Gell-Mann formula for any $sl(n, \mathbb{R})$ algebra w.r.t its $so(n)$ subalgebra takes the following form:

$$(22) \quad T_{ab}^{\sigma_2 \dots \sigma_n} = i \sum_{c>d}^n \{K_{cd}, D_{(cd)(ab)}^{\square\square}\} + i \sum_{c=2}^n \sigma_c D_{(cc)(ab)}^{\square\square},$$

where σ_c is a set of $n-1$ arbitrary parameters that essentially (up to some discrete parameters) label $sl(n, \mathbb{R})$ irreducible representations. Note that the sum in the first term goes only over pairs (c, d) where $c > d$ i.e. it is not symmetric in c, d .

Let us begin the proof that the expressions (22) satisfy the $sl(n, \mathbb{R})$ commutation relation (6) by introducing operators:

$$(23) \quad T_{ab}^{[c]} = i \sum_{d=1}^{c-1} \{K_{cd}, D_{(cd)(ab)}^{\square\square}\} + i \sigma_c D_{(cc)(ab)}^{\square\square}, \quad c = 2, \dots, n$$

i.e. expressing the generalized expression (22) as:

$$(24) \quad T_{ab} = \sum_{c=2}^n T_{ab}^{[c]}.$$

A straightforward, but somewhat lengthy, calculation yields $[T_{ab}^{(c)}, T_{a'b'}^{(d)}] = [T_{a'b'}^{[c]}, T_{ab}^{[d]}]$ for $c \neq d$, and thus we find:

$$(25) \quad [T_{ab}, T_{a'b'}] = \sum_c [T_{ab}^{[c]}, T_{a'b'}^{[c]}] = -i \sum_{c,d,d'} \{K_{dd'}, \{D_{(cd)(ab)}^{\square\square}, D_{(cd')(a'b')}^{\square\square}\}\}.$$

By making use of the identity:

$$(26) \quad \begin{aligned} & \sum_c (D_{(cd)(ab)}^{\square\square} D_{(cd')(a'b')}^{\square\square} - D_{(cd')(ab)}^{\square\square} D_{(cd)(a'b')}^{\square\square}) \\ &= \frac{1}{2} (\delta_{aa'} D_{(dd')(bb')}^{\square\square} + \delta_{bb'} D_{(dd')(aa')}^{\square\square} + \delta_{ab'} D_{(dd')(ba')}^{\square\square} + \delta_{ba'} D_{(dd')(ab')}^{\square\square}) \end{aligned}$$

and the fact that the M generators are given in terms of the K operators via the D^{\square} operators (cf. (20)), one verifies the desired expression (6).

Note that the first equality in (25) implies that the overall sign of operators $T_{ab}^{[c]}$ is inessential. Moreover, any left rotation (generated by the K operators) of the generalized formula (22) will preserve the $[T, T]$ commutator (6) and thus lead to another valid expression for the generalized Gell-Mann formula. The generalized formulas related in this way form an equivalence class of formulas that yield the same set of $sl(n, \mathbb{R})$ irreducible representations. Besides this class there are a few alternative useful expressions of the generalized Gell-Mann formula. We point out explicitly two cases below.

Let us consider operators:

$$(27) \quad U_{ab}^{(cd)} \equiv D_{(cd)(ab)}^{\square\square}$$

stressing that $D_{(cd)(ab)}^{\square\square}$ is just a particular representation of the U_{ab} operators (13), characterized by the choice of the vector v to be $v = (cd)$ and $|u| = 1$. Then, by making use of the commutation relations to shift the K operators to the right in (22) we find :

$$(28) \quad T_{ab} = 2i \sum_{c>d}^n U_{ab}^{(cd)} K_{cd} + i \sum_{c=2}^n \sigma'_c U_{ab}^{(cc)}.$$

This expression for the generalized formula can now be directly compared to the original formula in the form (12). It is as simple as the original Gell-Mann formula, with a crucial advantage of being valid in the whole representation space over $\mathcal{L}^2(Spin(n))$. General validity of the new formula is reflected in the fact that there are now $n - 1$ free parameters, i.e. representation labels, matching the $sl(n, \mathbb{R})$ algebra rank, compared to just one parameter of the original Gell-Mann formula.

Another notable form of the generalized formula relies on the fact that the operators $T^{[c]}$ (23) can be written as:

$$(29) \quad T_{ab}^{[c]} = \frac{i}{2} [C_2(so(c)_K), U_{ab}^{(cc)}] + i\sigma_c U_{ab}^{(cc)}, \quad c = 2, \dots, n$$

where $C_2(so(c)_K)$ is the second order Casimir of the $so(c)$ left action subalgebra, i.e. $C_2(so(c)_K) = \frac{1}{2} \sum_{a,b=1}^c (K_{ab})^2$. The generalized Gell-Mann formula can now be written as:

$$(30) \quad T_{ab}^{\sigma_2 \dots \sigma_n} = i \sum_{c=2}^n \frac{1}{2} [C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)},$$

which is to be compared with the original formula in the form (11). Again, the generalized formula matches, by simplicity of the expression, the original one. Besides, the very term when $c = n$ is, essentially, the original Gell-Mann formula (since $C_2(so(n)_K) = C_2(so(n)_M)$), whereas the rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some representations yielding the original formula.

The generalized Gell-Mann formula expression for the noncompact “shear” generators T_{ab} holds for all cases of $sl(n, \mathbb{R})$ irreducible representations, irrespective of their $so(n)$ subalgebra multiplicity (multiplicity free of the original Gell-Mann formula, and nontrivial multiplicity) and whether they are tensorial or spinorial. The price paid is that the Generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators K_{ab} and $U^{(cd)}_{ab}$.

5 Direct application - matrix elements of $SL(n, \mathbb{R})$ generators for all irreducible representations

The generalized Gell-Mann formula, as given by (30), can be directly applied to yield all matrix elements of the $\overline{SL}(n, \mathbb{R})$ generators for all irreducible representations, characterized by a complete set of labels σ_i , $i = 2, 3, \dots, n$ (the invariant Casimir operators are analytic functions of solely these labels), in the basis of the maximal compact subgroup $Spin(n)$. Note that there can be some additional discrete labels generally related to the finite $\overline{SL}(n, \mathbb{R})$ center group. Taking the matrix elements of (30) we get:

$$\begin{aligned}
& \left\langle \left(\begin{Bmatrix} J' \\ k' \end{Bmatrix} \right)_{\{m'\}} \left| T_{ab}^{\sigma_2 \dots \sigma_n} \right| \left(\begin{Bmatrix} J \\ k \end{Bmatrix} \right)_{\{m\}} \right\rangle \\
&= \left\langle \left(\begin{Bmatrix} J' \\ k' \end{Bmatrix} \right)_{\{m'\}} \left| i \sum_{c=2}^n \frac{1}{2} [C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)} \right| \left(\begin{Bmatrix} J \\ k \end{Bmatrix} \right)_{\{m\}} \right\rangle \\
&= \frac{i}{2} \sum_{c=2}^n (C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \sigma_c) \left\langle \left(\begin{Bmatrix} J' \\ k' \end{Bmatrix} \right)_{\{m'\}} \left| U_{ab}^{(cc)} \right| \left(\begin{Bmatrix} J \\ k \end{Bmatrix} \right)_{\{m\}} \right\rangle \\
&= \frac{i}{2} \sqrt{\frac{\dim(\{J\})}{\dim(\{J'\})}} \sum_{c=2}^n (C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \sigma_c) C_{\{k\}(cc)\{k'\}}^{\{J\}\square\square\{J'\}} C_{\{m\}(ab)\{m'\}}^{\{J\}\square\square\{J'\}},
\end{aligned}$$

where, in the last equality, the expression (17) for the matrix elements of the U operators is used. The second Clebsch-Gordan coefficient, that is merely reflecting the Wigner-Eckart theorem, can be evaluated in any suitable basis, not necessarily the Cartesian one, due to the fact that the expression is covariant with respect to the free index (ab) . Note, that this

is not the case for the first Clebsch-Gordan coefficient – it is necessary in order to evaluate it to express the specific vector $\left| \begin{smallmatrix} \square \square \\ (cc) \end{smallmatrix} \right\rangle$ in some basis that spans the entire vector space over $Spin(n)$.

The final expression is simplified by choosing the indexes of the generalized Gell-Mann formula matrix elements to be given by labels of the $Spin(n) \supset Spin(n-1) \supset \dots \supset Spin(2)$ group chain representation labels. In this notation, the basis vectors of the $Spin(n)$ irreducible representations are written as:

$$(32) \quad \left| \begin{smallmatrix} \{J\} \\ \{m\} \end{smallmatrix} \right\rangle = \left| \begin{array}{cccc} J_{Spin(n),1} & J_{Spin(n),2} & J_{Spin(n),3} & \dots \\ J_{Spin(n-1),1} & J_{Spin(n-1),2} & \dots & \\ & \dots & & \\ & & J_{Spin(2)} & \end{array} \right\rangle.$$

Likewise, the set of indices $\{k\}$ of (14) is thus given by the labels of the irreducible representations $\{J_{Spin(n-1),1}, J_{Spin(n-1),2}, \dots; J_{Spin(n-2),1}, J_{Spin(n-2),2}, \dots; \dots; J_{Spin(2)}\}$ of the $Spin(n) \supset Spin(n-1) \supset \dots \supset Spin(2)$ group chain.

To express the vector $\left| \begin{smallmatrix} \square \square \\ (cc) \end{smallmatrix} \right\rangle$ in such a basis we notice first that it corresponds to a diagonal traceless n by n matrix of the form $diag(-\frac{1}{n}, \dots, -\frac{1}{n}, \frac{n-1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n})$, with $\frac{n-1}{n}$ positioned at the c -th row and column. On the other hand, the diagonal traceless matrix $\sqrt{\frac{1}{c(c-1)}} diag(-1, \dots, -1, c-1, 0, \dots, 0)$, with first $c-1$ occurrences of -1 , corresponds to a vector that belongs to a second order symmetric tensor ($\square \square$ representation) with respect to $Spin(c), Spin(c+1), \dots, Spin(n)$ subgroups, and it is invariant under $Spin(c-1)$:

$$(33) \quad \left| \begin{array}{c} \{\square \square\}_{Spin(n)} \\ \dots \\ \{\square \square\}_{Spin(c)} \\ \{0\}_{Spin(c-1)} \\ \dots \\ 0 \end{array} \right\rangle.$$

This vector has $n-c+1$ double-boxes followed by $c-2$ zeros underneath – in shorthand notation: $\left| \begin{smallmatrix} \square \square \\ \{0\}^{c-2} \end{smallmatrix} \right\rangle^{n-c+1}$. Somewhat peculiar is the matrix $\sqrt{\frac{1}{2}} diag(-1, 1, 0, 0, \dots)$ that corresponds to:

$$(34) \quad \left| \begin{smallmatrix} \square \square \\ \{0\}^0 \end{smallmatrix} \right\rangle^{n-1} \equiv \frac{1}{\sqrt{2}} \left| \begin{array}{c} \{\square \square\}_{Spin(n)} \\ \dots \\ \{\square \square\}_{Spin(4)} \\ 2 \\ 2 \end{array} \right\rangle + \frac{1}{\sqrt{2}} \left| \begin{array}{c} \{\square \square\}_{Spin(n)} \\ \dots \\ \{\square \square\}_{Spin(4)} \\ 2 \\ -2 \end{array} \right\rangle,$$

where the standard labelling for $SO(n)$, $n \leq 3$ is implied, in particular the $\square\square$ representation corresponds to $J_{Spin(3)} = 2$.

By combining these facts we find:

$$(35) \quad \left| \begin{array}{c} \square\square \\ (cc) \end{array} \right\rangle + \frac{1}{c} \sum_{d=c+1}^n \left| \begin{array}{c} \square\square \\ (dd) \end{array} \right\rangle = \sqrt{\frac{c-1}{c}} \left| \begin{array}{c} \square\square \\ \{0\}^{c-2} \end{array} \right\rangle^{\{0\}^{n-c+1}}.$$

However, when evaluating the $U^{(cc)}$ operators of (30) in this basis, only the first term on the left-hand side is relevant due to the fact that:

$$(36) \quad d > c \quad \Rightarrow \quad [C_2(so(c)_K), U_{ab}^{(dd)}] = 0.$$

Having this in mind, we make use of (35) to recast, in the first equality of (31), the $U^{(cc)}$ operators accordingly. Taking into account arbitrariness of the σ_c coefficients and following the same steps as in (31), we finally obtain a rather simple expression for the shear generator matrix elements for an arbitrary $sl(n, \mathbb{R})$ representation (labelled now by parameters $\tilde{\sigma}_c$):

$$(37) \quad \left\langle \begin{array}{c} \{J'\} \\ \{k'\} \{m'\} \end{array} \middle| T_{\{w\}} \middle| \begin{array}{c} \{J\} \\ \{k\} \{m\} \end{array} \right\rangle = \sqrt{\frac{\dim(\{J\})}{\dim(\{J'\})}} C_{\{m\}\{w\}\{m'\}}^{\{J\}\square\square\{J'\}} \\ \times \sum_{c=2}^n \sqrt{\frac{c-1}{c}} \left(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c \right) C_{\{k\}}^{\{J\}\square\square_{(0)^{c-2}}\{J'\}}^{\{J\}\{m\}\{m'\}}.$$

The relation of the labelling of (37) and the one of (22), i.e. (30), is achieved provided $\sigma_c = \tilde{\sigma}_c + \sum_{d=2}^{c-1} \tilde{\sigma}_d/d$. The Clebsch-Gordan coefficient with indices $\{m\}, \{w\}, \{m'\}$ in (37) can be evaluated in an arbitrary basis (which is stressed by denoting the appropriate index by w instead by ab). The other Clebsch-Gordan coefficient can be evaluated in any basis labelled according to the $Spin(n) \supset Spin(n-1) \supset \dots \supset Spin(2)$ subgroup chain (e.g. Gel'fand-Tsetlin basis) and can be, nowadays, rather easily evaluated, at least numerically.

As already stated, the matrix elements of the $sl(n, \mathbb{R})/so(n)$ operators, as given by the Generalized Gell-Mann formula, apply to all tensorial, spinorial, unitary, nonunitary (both finite and infinite-dimensional) $sl(n, \mathbb{R})$ irreducible representations. In many physics applications one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present work, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(Spin(n), \kappa)$ of square integrable functions with a scalar product in terms of an arbitrary kernel κ , and to impose the unitarity constraints both on the scalar products itself and on the $sl(n, \mathbb{R})/so(n)$ operators matrix elements in that scalar product (cf. [18]).

To sum up, the expressions (16) and (37) fully determine the action of the $sl(n, \mathbb{R})$ operators for an arbitrary irreducible representation given by the set of $n - 1$ invariant Casimir operators labels $\tilde{\sigma}_c$. This action is given in terms of the basis vectors (14) of the representation spaces of the maximal compact subgroup $Spin(n)$ of the $\overline{SL}(n, \mathbb{R})$ group. This result is general due to a Corollary of Harish-Chandra [20] that explicitly applies to the case of the $sl(n, \mathbb{R})$ algebras.

6 Conclusion

The Gell-Mann formula, as stated above, applies at the pure algebraic level, i.e. as an algebraic expression, only in the case of (pseudo) orthogonal algebras. One can formally write it in other Lie algebra cases. However, it turns out, by an explicit verification, that it is not generally valid, i.e. that the closure of the commutation relations of the generators given by this formula is not granted generically. It turns out that certain successful applications of the Gell-Mann formula, beyond the orthogonal-like algebras, were actually carried out in a framework of particular algebra representation spaces. Thus, the Gell-Mann formula applicability can be broaden beyond the $so(m, n)$ algebra cases, by utilizing it in other algebra cases provided some Lie algebra representation conditions are met. As for the $sl(n, \mathbb{R})$ algebras, contracted w.r.t. their $so(n)$ subalgebras, the algebraic expression of the Gell-Mann formula matters generally for the multiplicity free representations only. In a previous work, starting from the known generic noncompact generators representation expressions for $sl(3, \mathbb{R})$ and $sl(4, \mathbb{R})$, we found the generalization of the Gell-Mann formula for the $sl(5, \mathbb{R})$ case. It is valid for all representation spaces irrespective of the $so(5)$ sub-representations non-trivial multiplicity. An analysis of the structure of the generalized Gell-Mann formula in the $sl(3, \mathbb{R})$, $sl(4, \mathbb{R})$ and $sl(5, \mathbb{R})$ cases, especially of the role played by the K_{ab} operators that generate the $SO(n)_K$, i.e. $Spin(n)_K$ group (acting to the left in the group manifold and characterize the $SO(n)_M$, i.e. $Spin(n)_M$ representations multiplicity) paved a way for a successful generalization of the Gell-Mann formula for all $sl(n, \mathbb{R})$ algebras. The generalized formula is given by expression (22). It can take alternative forms, such as (28) and (30), that suit better for certain mathematical or physical applications. The generalized Gell-Mann formula for the $sl(n, \mathbb{R})$ and or $su(n)$ algebras, considered w.r.t. their $so(n)$ subalgebras (the maximal compact subalgebra of the $sl(n, \mathbb{R})$ algebra) is compact and simple, and thus has a great potential for both further general consideration and various applications in mathematics and physics.

Note that the generalized Gell-Mann formula for the $sl(n, \mathbb{R})$ algebras, that is valid for all representation Hilbert spaces, is characterized precisely by a right number of $n - 1$ (algebra rank) parameters, i.e. representation labels. As a first and most precious application, based on the generalized formula, we obtained for the first time a closed form of the expressions of all matrix elements of the $sl(n, \mathbb{R})$ noncompact generators for all irreducible representations. All representations meaning: finite, infinite, tensorial and spinorial. A distinct feature of our generalized Gell-Mann formula approach is that the generalized expression goes beyond the standard notion of deformation of the contracted algebra, as it depends on additional operators, K_{ab} , not belonging, however directly related, to the contracted algebra. Due to this fact, our generalization of the Gell-Mann formula is remarkably simple (compared to complicated polynomial expressions appearing in some other approaches to generalize the Gell-Mann formula), nevertheless establishing a direct relation between representations of the contracted and original algebras.

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